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SHORT-TIME SOLUTION FOR UNSTEADY FORCED CONVECTION HEAT TRANSFER FROM AN IMPULSIVELY STARTED CIRCULAR CYLINDER

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Abstract—Short-time solution for unsteady heat transfer from an impulsively started circular cylinder is presented. Consideration is given to the case where unsteady temperature field is produced by the sudden imposition of a constant temperature difference between the body and the fluid as the impulsive motion is started. The present theory should be valid for any Prandtl number and for any Reynolds number larger than about 100. The Nusselt number results obtained are compared with the available numerical and theoretical ones.

NOMENCLATURE

h, heat-transfer coefficient;
H_1, H_2, functions defined by (53);
k, thermal conductivity;
Nu, local Nusselt number;
$\overline{Nu}$, mean Nusselt number;
Pe, Peclet number;
Pr, Prandtl number;
r, non-dimensional co-ordinate in radial direction;
r_0, radius of the cylinder;
R, radial co-ordinate defined by (8);
Re, Reynolds number;
t, non-dimensional temperature;
t', temperature;
T, non-dimensional time;
T_w, temperature on the surface;
T_∞, temperature at infinity;
t_n, temperature in the inner region, equation (18);
T_n, temperature in the outer region, equation (19);
u, non-dimensional velocity in r direction;
u', velocity in r direction;
U, free stream velocity;
v, non-dimensional velocity in θ direction;
v', velocity in θ direction.

Greek symbols

α, constant defined by (3);
ε, small parameter defined by (3);
η, variable defined by (12);
θ, co-ordinate in tangential direction;
ν, kinematic viscosity;
ξ, variable defined by (42);
τ, non-dimensional time;
τ', time;
τ_0, characteristic time;
ψ, non-dimensional stream function;
ψ_n^i, stream function in the inner region, equation (6);
ψ_n^0, stream function in the outer region, equation (7).

INTRODUCTION

THE PROBLEM of transient flow past an impulsively started circular cylinder has been studied by many authors. Blasius [1], Goldstein and Rosenhead [2], Schuh [3], Wundt [4], and Watson [5] have obtained short-time solutions in the limiting case of infinite Reynolds number. Wang [6], Collins and Dennis [7], and Bar-Lev and Yang [8] have extended the works to finite but large Reynolds number. Wang, applying the method of matched asymptotic expansions, obtained the series solution in powers of the normalized time up to the second term. Collins and Dennis, using a different formulation from that of Wang, obtained the series solution up to the seventh term, in which only the first approximations were derived analytically and the succeeding were computed numerically. Bar-Lev and Yang extended the work of Wang to the third order. Their formulation consists of only one governing nonlinear equation, that of vorticity; the troublesome pressure field is therefore eliminated. Their results agree well with those of Collins and Dennis. Many authors also gave numerical solutions of the Navier–Stokes equations.

The extensions of the above flow problem to transient heat transfer problems, however, have little attention in the literature. Recently, Jain and Goel [9] approached a heat-transfer problem numerically. They investigated the case in which unsteady thermal field is produced by sudden imposition of a constant temperature difference between the body and the fluid as the impulsive motion is strated and gave the numerical solutions for Pr (Prandtl number) = 0.73, Re (Reynolds number) = 50 and 100. However, short-time results for $T < 1$, T being the normalized time, are not given there.

The present paper reconsider this problem analytically, giving a short-time solution by extending the analysis of Bar-Lev and Yang for the velocity field. The results obtained in the present paper should be valid for any Prandtl number, $Re > 100$ and $T < 1$.

FORMULATION OF THE PROBLEM

Consider an unsteady flow of a viscous incompressible fluid past a circular cylinder of radius r_0 which is started impulsively from rest at time $\tau' = 0$ and subsequently moves with a constant velocity U in the direction of $\theta = 0$. Using the polar co-ordinate (r, θ) with origin at the center of the cylinder, the governing equation for the velocity field can be written in non-dimensional form as [8]

$$\left[\frac{\partial}{\partial \tau} + \frac{\varepsilon}{r}\left(\frac{\partial \psi}{\partial r}\frac{\partial}{\partial \theta} - \frac{\partial \psi}{\partial \theta}\frac{\partial}{\partial r}\right) - \varepsilon^2 \alpha \nabla^2\right]\nabla^2 \psi = 0, \tag{1}$$

where non-dimensional time τ is referred to an arbitrary characteristic time τ_0, radial co-ordinate r to r_0, the stream function ψ to $r_0 U$, and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{2}$$

$$\varepsilon = U\tau_0/r_0, \quad \alpha = (\varepsilon Re)^{-1}, \tag{3}$$

Re being the Reynolds number equal to Ur_0/ν (ν; kinematic viscosity). For the initial phase of the flow, ε is a small quantity. The stream function ψ is related to non-dimensional velocities $u = u'/U$ and $v = v'/U$, u' and v' being velocity components in r and θ directions respectively, by the relations

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v = \frac{\partial \psi}{\partial r}. \tag{4}$$

The initial and boundary conditions for ψ are

$$\left.\begin{array}{ll} \tau < 0 & \psi = 0, \\ \tau \geqslant 0 & \left\{\begin{array}{lll} \psi = \psi_r = 0 & \text{at} & r = 1, \\ \psi \to r\sin\theta & \text{as} & r \to \infty. \end{array}\right. \end{array}\right\} \tag{5}$$

Bar-Lev and Yang, assuming the Reynolds number to be large such that α is finite, obtained the solution for the initial flow field by applying the method of matched asymptotic expansions with ε as a small parameter. The method of solution is as follows. The velocity field is divided into two regions; one is the inner region close to the cylinder and the other is the outer region far from it. Separate, locally valid, expansions of the stream function are developed for these two regions. These "inner" and "outer" expansions are, respectively, assumed to be of the forms

$$\psi = \varepsilon\psi_1^i(R, \theta, \tau) + \varepsilon^2\psi_2^i(R, \theta, \tau) + \varepsilon^3\psi_3^i(R, \theta, \tau) + \ldots, \tag{6}$$

and

$$\psi = \psi_1^0(r, \theta, \tau) + \varepsilon\psi_2^0(r, \theta, \tau) + \varepsilon^2\psi_3^0(r, \theta, \tau) + \ldots, \tag{7}$$

where

$$R = (r-1)/\varepsilon. \tag{8}$$

Substitution of these expansions into (1) yields a set of equations for ψ_n^i and ψ_n^0. These equations can be successively solved by applying the matching condition that the two expansions must agree in the overlapping domain where both expansions are valid, as well as the physical boundary conditions (5). The final results thus obtained are as follows [8]:

$$\psi = (r - r^{-1})\sin\theta - [4/(\pi)^{1/2}](T/Re)^{1/2}r^{-1}\sin\theta - (T/Re)r^{-1}\sin\theta + 8\pi^{-1/2}[8/3(2)^{1/2} - 4/9\pi - 1]T(T/Re)^{1/2}r^{-2}\sin\theta\cos\theta + \ldots, \tag{9}$$

in the outer region and

$$\psi = 4(T/Re)^{1/2}\{\eta\,\mathrm{erf}\,\eta - \pi^{-1/2}[1 - \exp(-\eta^2)]\}\sin\theta + 4\pi^{-1/2}(T/Re)A\sin\theta + 8T(T/Re)^{1/2}B\sin\theta\cos\theta + \ldots, \tag{10}$$

in the inner region, where

$$T = \varepsilon\tau \equiv U\tau'/r_0, \tag{11}$$

τ' being the dimensional time,

$$\eta = R/[2(\alpha\tau)^{1/2}], \tag{12}$$

$$A = -\eta^2 + 2\eta - (\pi^{1/2}/4)(1 - \mathrm{erfc}\,\eta) + (3\pi^{1/2}/2)\eta^2\,\mathrm{erfc}\,\eta - (3/2)\eta\exp(-\eta^2) \tag{13}$$

$$B = (11/6\pi^{1/2})\exp(-\eta^2)\operatorname{erfc}\eta - [8/3(2\pi)^{1/2}]\operatorname{erfc}(2^{1/2}\eta)$$
$$+(1/3)\eta^3\operatorname{erfc}^2\eta - (2/3\pi^{1/2})\eta^2\exp(-\eta^2)\operatorname{erfc}\eta + (1/3\pi)\eta\exp(-2\eta^2)$$
$$-(1/2)\eta\operatorname{erfc}^2\eta + (4/9\pi - 3/2)\pi^{-1/2}\exp(-\eta^2)$$
$$-(1+4/9\pi)\eta^3\operatorname{erfc}\eta + (1+4/9\pi)\pi^{-1/2}\eta^2\exp(-\eta^2) + (2/3\pi^{1/2})\operatorname{erfc}\eta$$
$$+(1/2 - 2/3\pi)\eta\operatorname{erfc}\eta + [8/3(2)^{1/2} - 4/9\pi - 1]\pi^{-1/2}. \tag{14}$$

It is to be noted that outer flow is irrotational to all orders of ε. These solutions should be valid for $T < 1$, $Re > 100$, and may be stretched to $T \approx 1$ for higher Reynolds number [8].

In order to solve the energy equation for the temperature field, we shall use the procedure similar to that just described. The energy equation can be written as

$$\left[\frac{\partial}{\partial \tau} + \frac{\varepsilon}{r}\left(\frac{\partial \psi}{\partial r}\frac{\partial}{\partial \theta} - \frac{\partial \psi}{\partial \theta}\frac{\partial}{\partial r}\right) - \frac{\varepsilon^2 \alpha}{Pr}\nabla^2\right]t = 0, \tag{15}$$

where

$$t = (t' - T_\infty)/(T_w - T_\infty), \tag{16}$$

t' being the dimensional temperature and T_w, T_∞ the temperatures on the surface and at infinity, respectively. We assume that the cylinder and the fluid are initially at the same temperature and, as the impulsive motion is started, a constant temperature difference between the cylinder and the fluid is suddenly imposed. Then, the initial and the boundary conditions for t may be written as

$$\left.\begin{array}{ll} \tau < 0 & t = 0, \\ \tau \geq 0 & \begin{cases} t = 1 & \text{at} \quad r = 1, \\ t \to 0 & \text{as} \quad r \to \infty. \end{cases} \end{array}\right\} \tag{17}$$

The inner and outer expansions are, respectively, assumed to be of the forms as

$$t = t_1(R, \theta, \tau) + \varepsilon t_2(R, \theta, \tau) + \varepsilon^2 t_3(R, \theta, \tau) + \ldots, \tag{18}$$

and

$$t = T_1(r, \theta, \tau) + \varepsilon T_2(r, \theta, \tau) + \varepsilon^2 T_3(r, \theta, \tau) + \ldots. \tag{19}$$

Substituting (18) into (15), using the solution of Bar-Lev and Yang for ψ, and collecting the various powers of ε, the following set of equations for t_n's may be obtained.

$$t_{1\tau} - Pr^{-1}\alpha t_{1RR} = 0, \tag{20}$$

$$t_{2\tau} - Pr^{-1}\alpha t_{2RR} = Pr^{-1}\alpha t_{1R} + \psi^i_{1\theta} t_{1R} - \psi^i_{1R} t_{1\theta}, \tag{21}$$

$$t_{3\tau} - Pr^{-1}\alpha t_{3RR} = Pr^{-1}\alpha(t_{2R} - Rt_{1R} + t_{1\theta\theta}) - \psi^i_{2R} t_{1\theta} - \psi^i_{1R} t_{2\theta}$$
$$+ \psi^i_{1\theta} t_{2R} + \psi^i_{2\theta} t_{1R} + R(\psi^i_{1R} t_{1\theta} - \psi^i_{1\theta} t_{1R}). \tag{22}$$

Similarly, substitution of (19) into (15) yields a set of the equations for T_n's. It is easy to show that the solutions of the resulting equations for T_n are

$$T_n = 0 \quad \text{for all } n, \tag{23}$$

so that the matching condition for the inner expansion (18) may be written as

$$t_n(R, \theta, \tau) \to 0(\text{exponentially}) \quad \text{as} \quad R \to \infty. \tag{24}$$

CONSTRUCTION OF THE SOLUTION

The solution of (20) subject to the boundary condition on the surface and the matching condition (24) is

$$t_1 = \operatorname{erfc}(Pr^{1/2}\eta), \tag{25}$$

where η is given by (12). Substituting this into (21), we have

$$t_{2\tau} - Pr^{-1}\alpha t_{2RR} = -(Pr/\pi)^{1/2}(Pr^{-1}(\alpha/\tau)^{1/2} + 4\{\eta\operatorname{erf}\eta - \pi^{-1/2}[1 - \exp(-\eta^2)]\}\cos\theta)\exp(-Pr\eta^2). \tag{26}$$

By putting

$$t_2 = (\alpha\tau)^{1/2} f_2(\eta) + \tau g_2(\eta)\cos\theta, \tag{27}$$

(26) becomes

$$f_2'' + 2Pr\eta f_2' - 2Prf_2 = 4(Pr/\pi)^{1/2}\exp(-Pr\eta^2), \tag{28}$$

$$g_2'' + 2Pr\eta g_2' - 4Prg_2 = 16(Pr/\pi)^{1/2}Pr\{\eta\operatorname{erf}\eta - \pi^{-1/2}[1 - \exp(-\eta^2)]\} \times \exp(-Pr\eta^2), \tag{29}$$

where prime denotes differentiation with respect to η. The required solutions of these equations are

$$f_2 = -\eta\,\mathrm{erfc}(Pr^{1/2}\eta), \tag{30}$$

and

$$\begin{aligned} g_2 = & -[4/\pi(2\pi)^{1/2}][Pr^{1/2}(Pr+5/3)-(Pr+1)^2\,\mathrm{arctg}\,Pr^{-1/2}] \\ & \times Hh_2[(2Pr)^{1/2}\eta]-\pi^{-1}(Pr+1)^2\,\mathrm{arctg}\,Pr^{-1/2}(2Pr\eta^2+1) \\ & +(Pr^{1/2}/\pi)\Big[\pi^{1/2}(Pr^2+2Pr-1)\eta\exp(-Pr\eta^2)\,\mathrm{erf}\,\eta \\ & +(Pr-1)\exp[-(Pr+1)\eta^2]+(8/3)\exp(-Pr\eta^2) \\ & +\pi^{1/2}(Pr+1)^2(2Pr\eta^2+1)\int_0^\eta \exp(-Pr\eta^2)\,\mathrm{erf}\,\eta\,\mathrm{d}\eta\Big], \end{aligned} \tag{31}$$

where $Hh_n(x)$ is a function given by [10]

$$Hh_n(x) = \begin{cases} \displaystyle\int_0^\infty \frac{t^n}{n!}\exp\{-\tfrac{1}{2}(t+x)^2\}\,\mathrm{d}t = \int_x^\infty \frac{(u-x)^n}{n!}\exp(-\tfrac{1}{2}u^2)\,\mathrm{d}u \quad \text{for } n \text{ an integer} \geqslant 0, & (32) \\ (-1)^{n-1}(\mathrm{d}/\mathrm{d}x)^{-n-1}\exp(-\tfrac{1}{2}x^2) \quad \text{for } n \text{ a negative integer}, & (33) \end{cases}$$

and satisfies the equation $\mathrm{d}^2y/\mathrm{d}x^2+x\cdot\mathrm{d}y/\mathrm{d}x-ny=0$.

Substitution of (25), (30), and (31) into (22) yields the following equation for t_3.

$$\begin{aligned} t_{3\tau}-Pr^{-1}\alpha t_{3RR} = {} & 2Pr^{-1}\alpha f_2'(\eta)+[2\alpha/(\pi Pr)^{1/2}]\eta\exp(-Pr\eta^2) \\ & +(\alpha\tau)^{1/2}\big((1/2Pr)g_2'(\eta)+2\{\eta\,\mathrm{erf}\,\eta-\pi^{-1/2}[1-\exp(-\eta^2)]\} \\ & \times[f_2'(\eta)+4(Pr/\pi)^{1/2}\eta\exp(-Pr\eta^2)]-[4(Pr)^{1/2}/\pi]\exp(-Pr\eta^2)A\big)\cos\theta \\ & +\tau\big(2g_2(\eta)\,\mathrm{erf}\,\eta\sin^2\theta+2\{\eta\,\mathrm{erf}\,\eta-\pi^{-1/2}[1-\exp(-\eta^2)]\}g_2'(\eta)\cos^2\theta \\ & -8(Pr/\pi)^{1/2}\exp(-Pr\eta^2)B\cos 2\theta\big). \end{aligned} \tag{34}$$

By putting

$$t_3 = \alpha\tau f_3(\eta)+\tau(\alpha\tau)^{1/2}g_3(\eta)\cos\theta+\tau^2h_3(\eta), \tag{35}$$

(34) becomes

$$f_3''+2Pr\eta f_3'-4Prf_3 = 2(2/\pi)^{1/2}\{2Hh_2[(2Pr)^{1/2}\eta]-2Pr\eta^2Hh_0[(2Pr)^{1/2}\eta]-2Hh_{-2}[(2Pr)^{1/2}\eta]\}, \tag{36}$$

$$\begin{aligned} g_3''+2Pr\eta g_3'-6Prg_3 = {} & -2g_2'(\eta)+8Pr\{-\eta\,\mathrm{erf}\,\eta+\pi^{-1/2}[1-\exp(-\eta^2)]\} \\ & \times[f_2'(\eta)+4(Pr/\pi)^{1/2}\eta\exp(-Pr\eta^2)]+[16Pr(Pr)^{1/2}/\pi]\exp(-Pr\eta^2)A, \end{aligned} \tag{37}$$

$$\begin{aligned} h_3''+2Pr\eta h_3'-8Prh_3 = {} & -8Pr\big[[g_2(\eta)\,\mathrm{erf}\,\eta+4(Pr/\pi)^{1/2}B\exp(-Pr\eta^2)]\sin^2\theta \\ & +(\{\eta\,\mathrm{erf}\,\eta-\pi^{-1/2}[1-\exp(-\eta^2)]\}g_2'(\eta)-4(Pr/\pi)^{1/2}B\exp(-Pr\eta^2))\cos^2\theta\big]. \end{aligned} \tag{38}$$

The solution of (36) is

$$\begin{aligned} f_3 &= [(2/\pi)^{1/2}/Pr]\{Hh_2[(2Pr)^{1/2}\eta]-(1/2)Hh_0[(2Pr)^{1/2}\eta]+(3/4)Hh_{-2}[(2Pr)^{1/2}\eta]\} \\ &= \mathrm{erfc}(Pr^{1/2}\eta)+[1/2(\pi Pr)^{1/2}]\eta\exp(-Pr\eta^2). \end{aligned} \tag{39}$$

The solution of (37) is also obtained, but its detailed expression is too lengthy to be reproduced here and is omitted. Only the surface derivative, $g_3'(0)$, will be shown in the next section.

In theory, an exact solution for h_3 can be found but it is too complicated to obtain it. It is, however, easy to obtain asymptotic solutions of h_3 for large and small Prandtl numbers. Since the surface heat transfer is of greater practical importance than the fluid temperature, attention is given here only to the calculation of $h_3'(0)$. It is easily shown that $h_3'(0)$ is given by

$$h_3'(0) = -[8(2)^{1/2}/(\pi)^{1/2}]\int_0^\infty R(\eta,\theta)\exp(Pr\eta^2)Hh_4[(2Pr)^{1/2}\eta]\,\mathrm{d}\eta, \tag{40}$$

where $R(\eta,\theta)$ denotes the right side of (38). We first consider the case of $Pr\to 0$. By introducing a new variable

$$\xi = Pr^{1/2}\eta, \tag{41}$$

and expanding the integrand in (40) in terms of $Pr^{1/2}$ with ξ fixed, we can show that $h_3'(0)$ is expressed as

$$h_3'(0) = -\frac{1}{3\pi(\pi)^{1/2}}\left\{4\pi Pr^{1/2}+\frac{64}{15}\left[9-16(2)^{1/2}+\frac{16}{3\pi}\right]Pr+\frac{64}{3}Pr^2+\frac{64}{15}Pr^3+\ldots\right\}\sin^2\theta$$
$$-\frac{1}{3\pi(\pi)^{1/2}}\left\{\pi Pr^{1/2}-256\left[\frac{11}{40}-\frac{4(2)^{1/2}}{15}+\frac{4}{45\pi}\right]Pr+1024\left(\frac{2}{9\pi}-\frac{5}{96}\right)Pr^{3/2}\right.$$
$$\left.-\frac{32}{3}Pr^2+\frac{512}{3}\left(\frac{4}{3\pi}-\frac{1}{4}\right)Pr^{5/2}-\frac{32}{15}Pr^3+\frac{512}{15}\left(\frac{4}{3\pi}-\frac{1}{4}\right)Pr^{7/2}+\ldots\right\}\cos^2\theta. \qquad (42)$$

This solution is valid for small Prandtl number fluids such as liquid metals. For $Pr \to \infty$, on the other hand, expanding the integrand in (40) in terms of $Pr^{-1/2}$ with ξ fixed, we can finally obtain the following equation for $h_3'(0)$,

$$h_3'(0) = \frac{128}{\pi(\pi)^{1/2}}\left[\frac{2}{105}\left(1+\frac{4}{3\pi}\right)+\frac{1}{3}\left(\frac{19}{96}-\frac{\pi}{64}-\frac{32}{45\pi}\right)Pr^{-1/2}+\frac{4}{315}\left(\frac{1}{6}+\frac{2}{9\pi}\right)Pr^{-1}\right.$$
$$+\left(\frac{33}{160}-\frac{992}{1575\pi}\right)Pr^{-3/2}-\frac{4}{10\,395}\left(\frac{1}{2}+\frac{2}{3\pi}\right)Pr^{-2}$$
$$\left.+\left(\frac{581}{1440}-\frac{18\,112}{14\,175\pi}\right)Pr^{-5/2}+\ldots\right]\sin^2\theta+\frac{128}{\pi(\pi)^{1/2}}\left[-\frac{2}{105}\left(1+\frac{4}{3\pi}\right)\right.$$
$$+\frac{1}{6}\left(-\frac{32}{25\pi}+\frac{33}{80}+\frac{\pi}{32}\right)Pr^{-1/2}-\frac{4}{315}\left(\frac{1}{6}+\frac{2}{9\pi}\right)Pr^{-1}+\frac{1}{315}\left(\frac{16\,141}{512}-\frac{1504}{15\pi}\right)Pr^{-3/2}$$
$$\left.+\frac{4}{10\,395}\left(\frac{1}{2}+\frac{2}{3\pi}\right)Pr^{-2}+\left(\frac{121}{840}-\frac{8912}{19\,845\pi}\right)Pr^{-5/2}+\ldots\right]\cos^2\theta. \qquad (43)$$

For moderate values of Pr, the calculation of $h_3'(0)$ is performed numerically and the results are shown in tabular form in Table 1 together with the numerical data evaluated from (42) and (43). It is seen from the Table that the expansion for large Pr, (43), is valid for $Pr \geqslant 5$.

DISCUSSION

We can now obtain the expression for the surface heat transfer from the solutions just obtained. Defining the local Nusselt number Nu as $Nu = hr_0/k$, where h is the heat-transfer coefficient $[= -(k/r_0)(\partial t/\partial r)_{r=1}]$, then we have

$$Nu = (Pe/T)^{1/2}\{1/(\pi)^{1/2}+(1/2)(T/Pe)^{1/2}+[2/\pi(\pi)^{1/2}][(Pr+1)^2\,\mathrm{arctg}\,Pr^{-1/2}$$
$$-(Pr)^{1/2}(Pr+5/3)]T\cos\theta-[1/4(\pi)^{1/2}](T/Pe)-(1/2)g_3'(0)[T(T/Pe)^{1/2}]\cos\theta$$
$$-[1/2(Pr)^{1/2}]h_3'(0)T^2+\ldots\}, \qquad (44)$$

where

$$g_3'(0) = \begin{cases} (Pr^{1/2}/30\pi)(300Pr^6+230Pr^5-410Pr^4-1639Pr^3-281Pr^2+477Pr \\ -197)/(1-Pr)(5Pr^3+3Pr^2+1)+(Pr+1)^2(20Pr^5-18Pr^4-8Pr^3-5Pr^2 \\ +20Pr-5)\,\mathrm{arctg}\,Pr^{-1/2}/2\pi(Pr-1)(5Pr^3+3Pr^2+1) \\ +(1/2)(4Pr^{1/2}/(1-Pr)+1/\pi^{1/2}-3/2) \quad \text{for} \quad Pr \neq 1, & (45) \\ -3/4+8/5\pi+1/2\pi^{1/2} \quad \text{for} \quad Pr = 1, & (46) \end{cases}$$

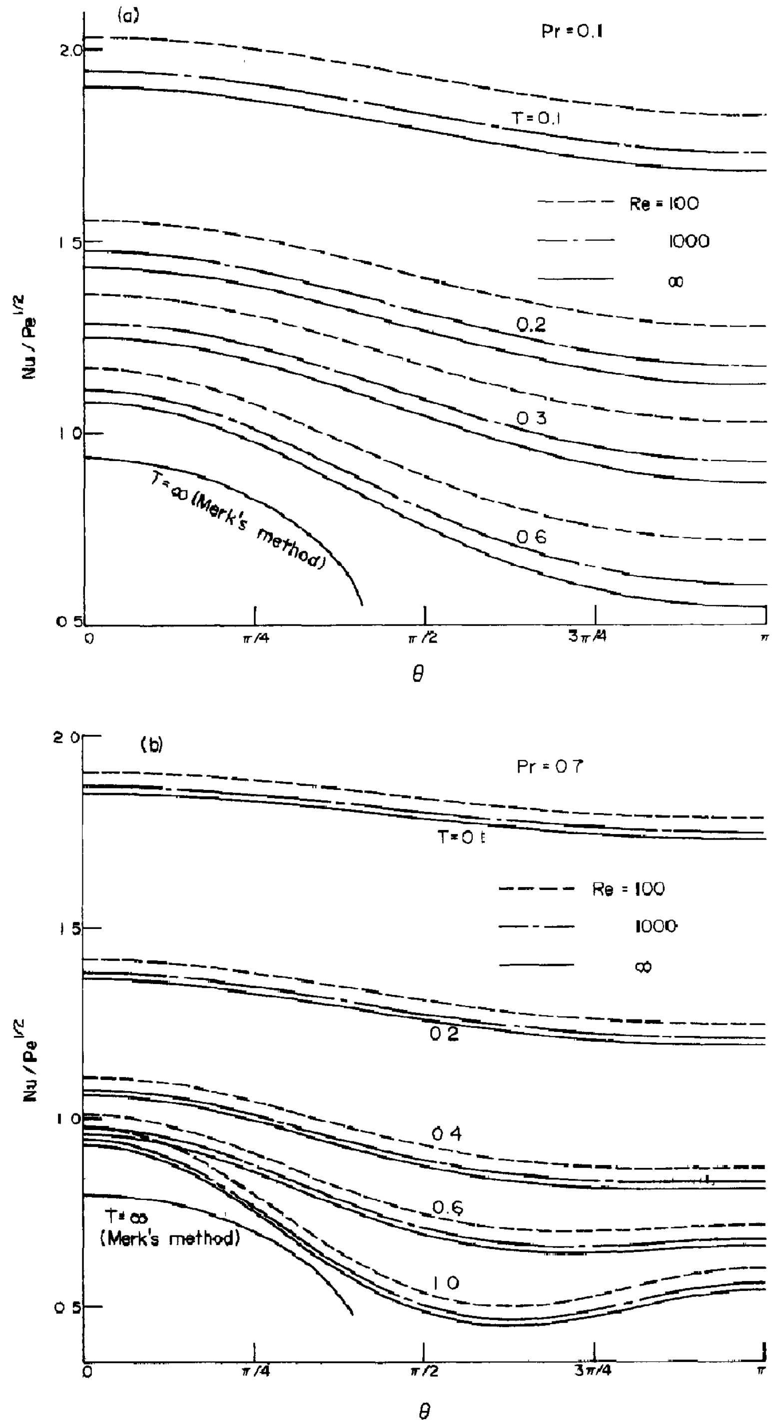
Table 1. Numerical values of $h_3'(0)$*

	$H_1(Pr)$		$H_2(Pr)$	
Pr	Numerical sol.	Asymptotic sol.	Numerical sol.	Asymptotic sol.
0.01	−0.0312	−0.0312	−0.0449	−0.0449
0.1	−0.0375		−0.115	
0.7	0.134		−0.287	
0.73	0.139		−0.292	
1	0.181		−0.324	
5	0.382	0.382	−0.463	−0.462
10	0.447	0.447	−0.506	−0.506

*$H_1(Pr)$ and $H_2(Pr)$ are related to $h_3'(0)$ by $h_3'(0) = H_1(Pr)\sin^2\theta+H_2(Pr)\cos^2\theta$.

and $h_3'(0)$ is given by (42), (43), and Table 1. The square-root singularity in time in (44) is obviously due to a discontinuity in the temperature existing on the surface at $T = 0$.

The local Nusselt number distributions around the cylinder at $Re = 100$, 1000, and ∞ are shown graphically in Figs. 1(a)–(c) for $Pr = 0.1$, 0.7, and 10. It is seen that at early times the Nusselt number decreases monotonously from the front stagnation point and takes its minimum value at the rear stagnation point, but that, except for the case of $Pr = 0.1$, the point of minimum Nusselt number ($\theta = \theta_c$) moves upstream in due time and the Nusselt number begins to increase at this point. This increase in Nusselt number may be considered to be closely related to the phenomena of flow separation. According to Bar-Lev and Yang, separations start at some values of T between about 0.32 and 0.4 and back flow occurs in the separated region. Like the point of minimum Nusselt number, the separation point ($\theta = \theta_s$) is first at $\theta = 180°$ and moves upstream in due time. In Fig. 2, we compare the progressions with time of θ_c and θ_s. It is seen from the figure that the influence of the Reynolds number on θ_c is much smaller than that on θ_s. The influence of the Prandtl number on θ_c, on the other hand, is much larger than that of the Reynolds number; θ_c shifts upstream with the increase of Pr, tending asymptotically to a constant value depending on T and Re. This suggests that the effect of flow separation on heat transfer is stronger for larger values of Pr. It is interesting to note that, above some critical value of Pr (depending on Re), the increase in Nusselt number can be observed before flow separates and, after flow separation occurs, the



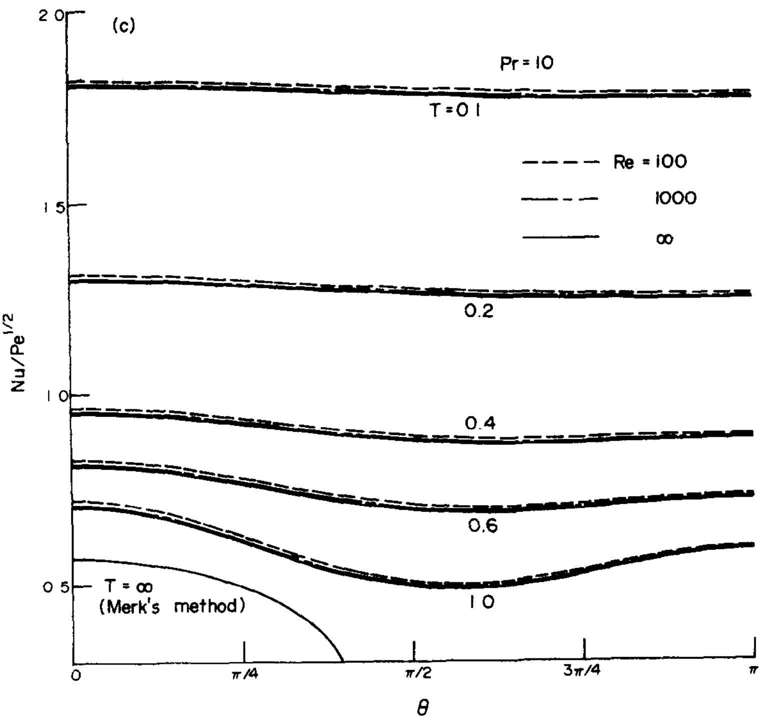


FIG. 1. Local Nusselt number distributions.

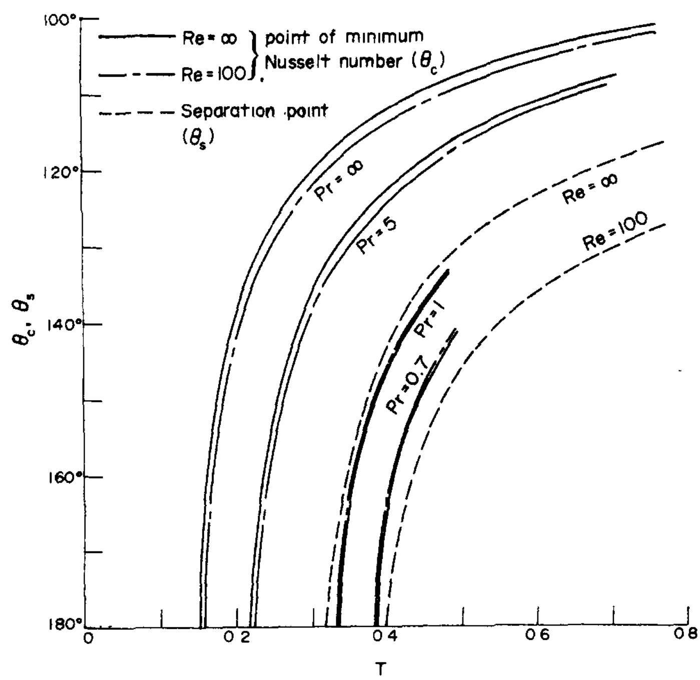


FIG. 2. Progressions with time of θ_c and θ_s.

increase of Nu begins in the unseparated region in front of the separation point. With the decrease of the Prandtl number, the effect of separation on heat transfer becomes small, that is, the increase in Nu occurs after flow separates and the point of minimum Nusselt number becomes to exist within the separated region. For $Pr = 0.1$, the increase of Nu cannot be observed in the range $T \leqslant 1$, as can be seen from Fig. 1. The reason why the effect of flow separation on heat transfer is small for low Prandtl number fluids may be explained as follows. For $T \leqslant 1$, the thickness of the back flow region is very thin [11] and hence, when Pr is very small, this region is confined to a very thin layer at the bottom of the thermal boundary layer because of the high thermal conductivity of the fluid. Therefore, the existence of the back flow region have little influence on heat transfer.

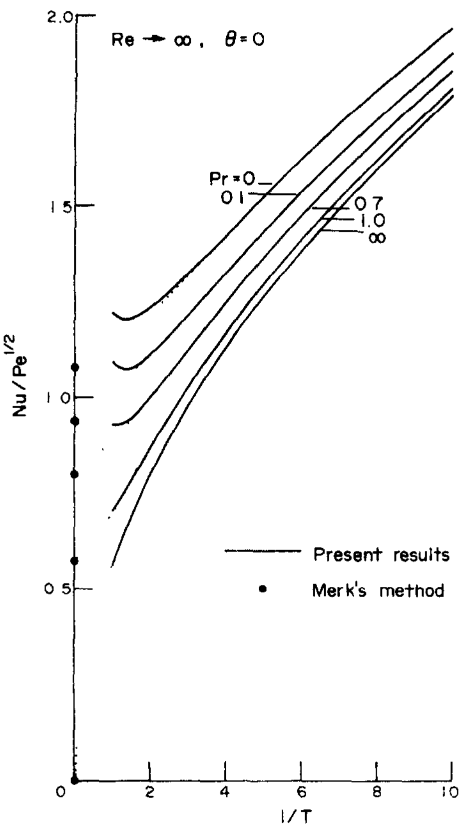


FIG. 3. Variation in local Nusselt number at the stagnation point for $Re \to \infty$.

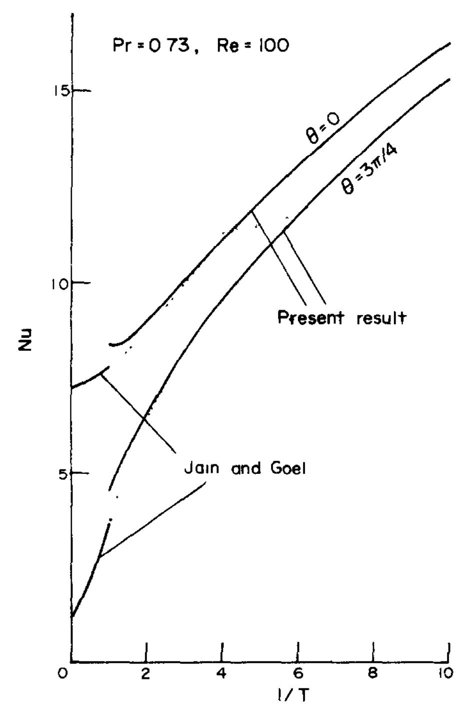


FIG. 4. Comparison of the present results for the local Nusselt number with those of Jain and Goel at $Pr = 0.73$ and $Re = 100$.

For comparison, the steady-state results for $Re \to \infty$ calculated by Merk's method [12], which is one of the most reliable methods for calculating steady heat transfer through laminar boundary layer, are also shown in Fig. 1 in the range from stagnation point to separation point. In applying Merk's method, the velocity distribution outside the boundary layer was taken to be [13]

$$U_p/U = 3.640(x'/D) - 3.20(x'/D)^3. \quad (47)$$

It seems from the comparison that in all cases, the present results up to $T = 0.6$ approach smoothly the steady-state curves, but that, except for the case of $Pr = 10$, the results for $T = 1$ is not reliable. This can be seen more clearly in Fig. 3, in which the local Nusselt number results are plotted as a function of time for $\theta = 0$ together with the steady-state results. It is seen that as the value of Pr becomes larger, the present results approach the steady-state values more smoothly, suggesting that the convergency of the expansion (44) becomes better for larger values of Pr. In Fig. 4, the present results for $T \leqslant 1$ are compared at $Pr = 0.73$ and $Re = 100$ with the numerical results of Jain and Goel for $T \geqslant 1$.

For the finite values of Pr, there exist no available data of the local Nusselt number for $T < 1$ to be compared with the present results. Only at $T = 1$, for which the expansion (44) is not expected to be valid, we can compare the present result with that of Jain and Goel at $Pr = 0.73$ and $Re = 100$. The comparison is shown in Fig. 5; the agreement is not satisfactory. In the limit of $Pr \to 0$, on the other hand, there exists, for $Pe \to \infty$, an analytical calculation to be compared with the present result. The calculation has been made by the present writer [14] using a potential flow approximation. It is well known that this approximation is valid for $Pr \to 0$ in the unseparated region from forward stagnation point to separation point. For $T < 1$ (and $Pr \to 0$), however, the potential flow approximation is valid in the separated region also, since the back flow region is so thin for $T < 1$ that the region is confined to a very thin layer at the bottom of the thermal layer when Pr is small, as is stated before, and the velocity field in the thermal layer may be essentially expressed by the inviscid flow. Thus, for $Pr \to 0$ and $T < 1$, it can be said that the velocity in the thermal boundary layer may be reasonably expressed by the inviscid flow outside the momentum layer over the whole region from forward to rear stagnation point. For $Re \to \infty$, the displacement effect of the inner flow is negligible, so that the velocity distribution of the inviscid flow is that around a circular cylinder of unit radius. For this velocity distribution, the present writer gave an analytical solution of the energy equation assuming the infinite Peclet number. Comparison of the present results for $Pr \to 0$ and $Pe \to \infty$ with those from the inviscid theory is shown in Figs. 6 and 7 together with the steady-state results by Merk's method. For $T \leqslant 0.4$, excellent agreement may be found between the present and the inviscid

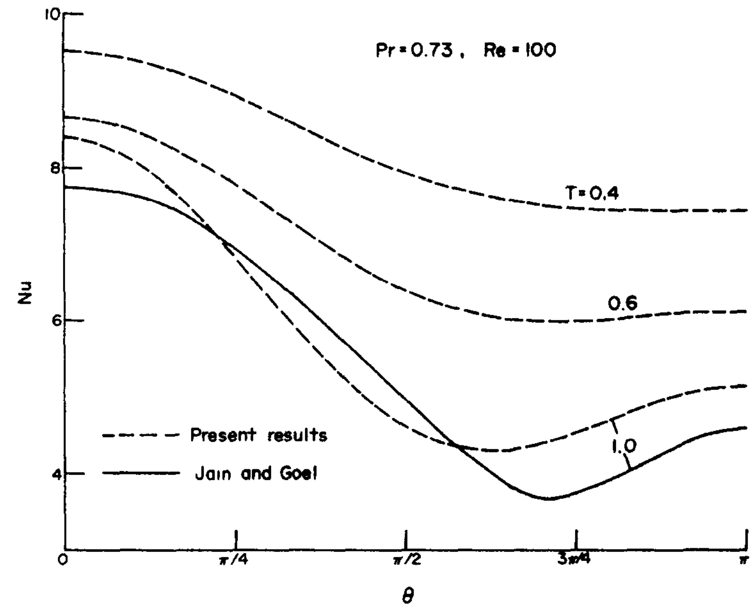


FIG. 5. Comparison of the present result for the local Nusselt number distribution with that of Jain and Goel at $T = 1$, $Pr = 0.73$, and $Re = 100$.

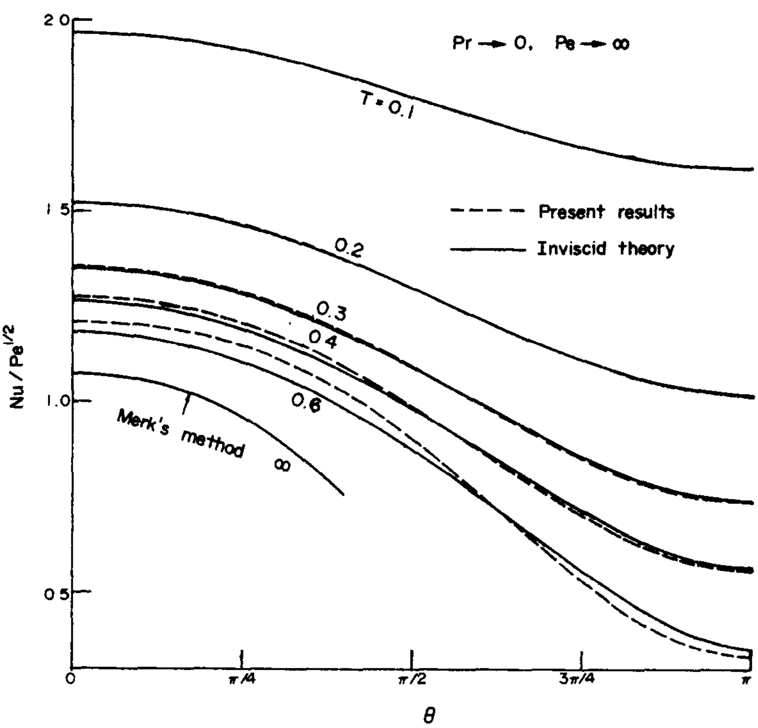


FIG. 6. Comparison of the present results for $Pr \to 0$ with those of inviscid theory.

results. For $T > 0.6$, the difference between them rapidly increases, especially on the front side of the cylinder.

For engineering applications, we are often concerned with the ratio of the instantaneous wall flux to its steady value, i.e. Nu/Nu_s, where Nu_s denotes the steady value of Nu. There are, however, no reliable methods for predicting Nu_s at an arbitrary point around a circular cylinder. Only in the upstream region of the cylinder where no separation occurs, we can calculate Nu_s for $Re \to \infty$ with sufficient accuracy by using Merk's method mentioned earlier. Therefore, the calculations of the ratio Nu/Nu_s have been made only for $Re \to \infty$ and only in the upstream region of the cylinder. The results for $\theta = 0$ and $\pi/4$ are shown in Fig. 8. It is seen from the

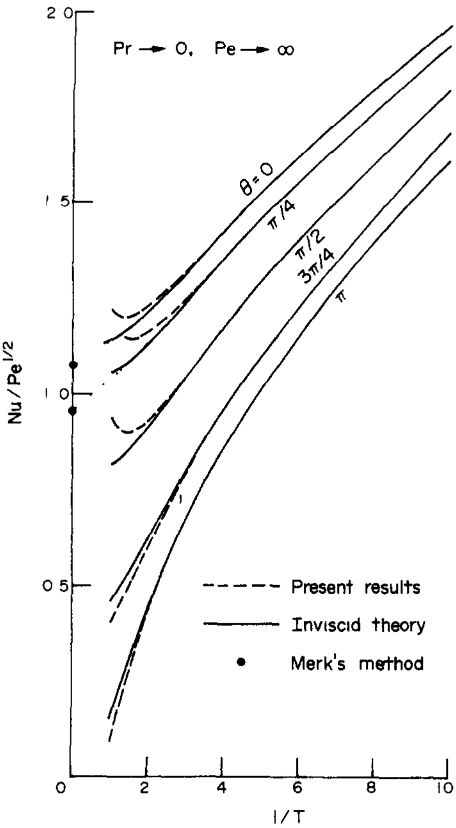


FIG. 7. Comparison of the present results for $Pr \to 0$ with those of inviscid theory.

figure that the response time of heat transfer increases monotonically with increasing Prandtl number.

Finally, we shall calculate the mean Nusselt number averaged over the surface of the cylinder; i.e.

$$\overline{Nu} = (1/\pi)\int_0^\pi Nu\,d\theta. \quad (48)$$

Using (44), the expression for $\overline{Nu}$ may be written as

$$\overline{Nu} = (Pe/T)^{1/2}\{1/(\pi)^{1/2} + (1/2)(T/Pe)^{1/2} - [1/4(\pi)^{1/2}](T/Pe) - [1/4(Pr)^{1/2}]h_3'(0)T^2 + \ldots\}. \quad (49)$$

Figure 9 shows the resulting curves of $\overline{Nu}$ plotted as a function of T at $Pr = 0.7$ and 10, respectively, for parametric values of Re and Fig. 10 shows those at $Re = \infty$ for parametric values of Pr. In Fig. 10, the result of the inviscid theory is also included for comparison.

CONCLUSION

The analysis of Bar-Lev and Yang for the problem of transient flow past an impulsively started circular cylinder is extended to analyse a transient temperature field which is produced by sudden imposition of a constant temperature difference between the cylinder and the fluid as the impulsive motion is started.

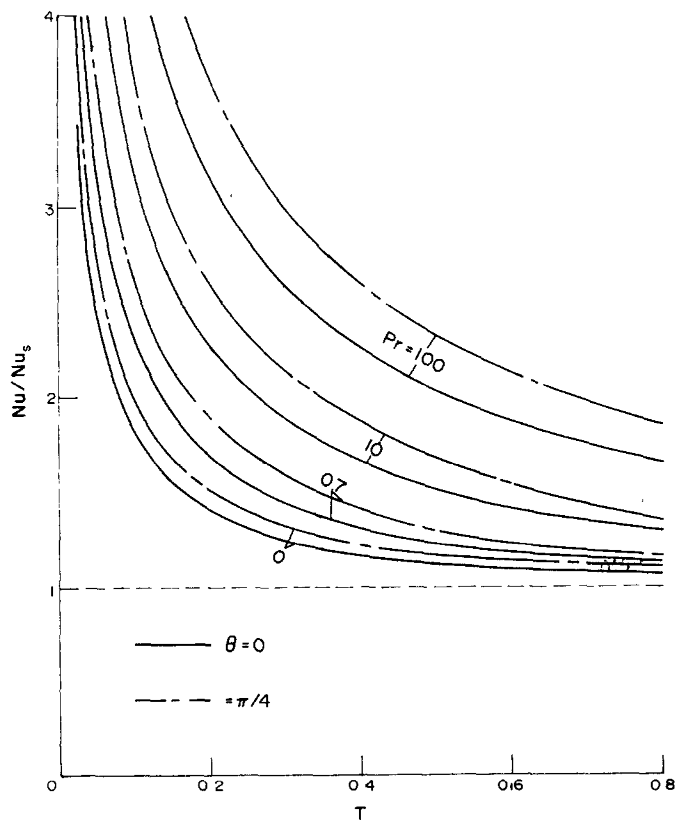


FIG. 8. Variation in the ratio Nu/Nu_s.

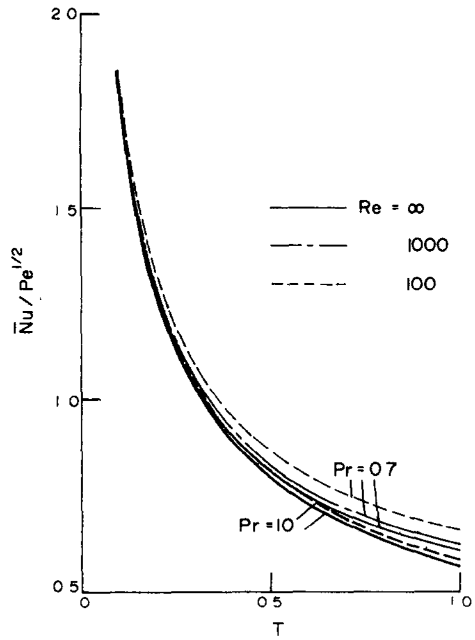


FIG. 9. Variation in the mean Nusselt number.

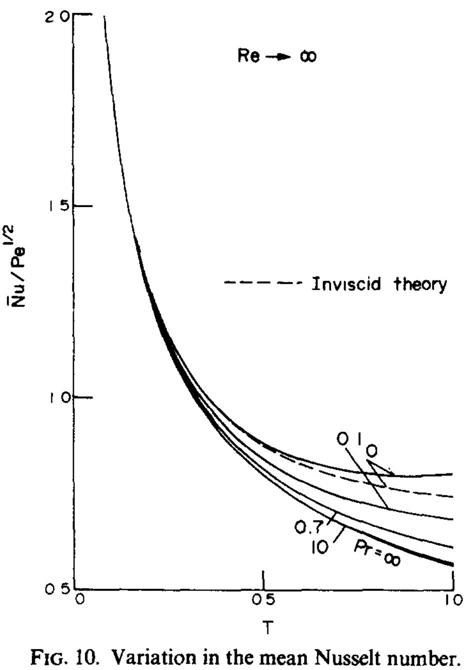


FIG. 10. Variation in the mean Nusselt number.

The main results obtained in this paper may be written as follows.

(1) The local Nusselt number and the mean Nusselt number averaged over the surface of the cylinder are given by (44) and (49), respectively, and their convergences become better for larger values of *Pr*.

(2) At early times after the impulsive start, the local Nusselt number along the surface decreases monotonously from the front stagnation point and takes its minimum value at the rear stagnation point. Except for the case of small Prandtl number, the point of minimum Nusselt number moves upstream in due time and the Nusselt number begins to increase at this point. This increase in *Nu* is more noticeable for larger values of *Pr*. For sufficiently small Prandtl number fluids, such an increase is not found for $T < 1$.

(3) Since the above increase in Nusselt number may be considered to be closely related to flow separation, comparison is made of the progression with time of the point of minimum Nusselt number and the point of separation. It is found that above some critical Prandtl number, the increase in *Nu* occurs before flow separates and after flow separation occurs, the increase in *Nu* begins in the unseparated region in front of the separation point. For smaller values of *Pr*, on the other hand, the increase in *Nu* begins after flow separates and the minimum Nusselt number takes place somewhere between the separation point and the rear stagnation point.

(4) The larger the Prandtl number is, the longer the response time of heat transfer is.

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SOLUTION DE TEMPS BREF POUR LA CONVECTION THERMIQUE FORCEE ET INSTATIONNAIRE AUTOUR D'UN CYLINDRE CIRCULAIRE EN MOUVEMENT IMPULSIONNEL

Résumé—On présente une solution de temps bref pour le transfert thermique instationnaire d'un cylindre circulaire en mouvement impulsionnel. On considère le cas où le champ de température variable est produit par l'imposition d'une différence de température constante entre le corps et le fluide lorsque le mouvement impulsionnel est déclanché. La présente théorie serait valable pour un nombre de Prandtl quelconque et pour un nombre de Reynolds supérieur à 100. Les nombres de Nusselt obtenus sont comparés avec ceux déjà connus numériquement ou théoriquement.

KURZZEIT-LÖSUNG FÜR INSTATIONÄRE WÄRMEÜBERTRAGUNG DURCH ERZWUNGENE KONVEKTION AN EINEM IMPULSARTIG BEWEGTEN KREISZYLINDER

Zusammenfassung—Es wird eine Kurzzeit-Lösung für die instationäre Wärmeübertragung an einem impulsartig bewegten Kreiszylinder angegeben. Es wird der Fall betrachtet, bei dem ein instationäres Temperaturfeld dadurch entsteht, daß plötzlich ein konstanter Temperaturunterschied zwischen Körper und Fluid aufgeprägt wird, sobald die impulsartige Bewegung beginnt. Die vorliegende Theorie müßte für alle Prandtl-Zahlen und für Reynolds-Zahlen größer als 100 gültig sein. Die in Nusselt-Zahlen ausgedrückten Ergebnisse werden mit den vorhandenen numerischen und theoretischen Werten verglichen.

РЕШЕНИЕ ЗАДАЧИ О НЕСТАЦИОНАРНОМ ТЕПЛООБМЕНЕ ПРИ ВЫНУЖДЕННОЙ КОНВЕКЦИИ ОТ КРУГЛОГО ЦИЛИНДРА, ИМПУЛЬСНО ПРИВЕДЕННОГО В ДВИЖЕНИЕ

Аннотация — Представлено решение задачи нестационарного теплообмена круглого цилиндра, импульсно приведенного в движение. Рассматривается случай, когда нестационарное температурное поле образуется при внезапном наложении постоянной разности температур между цилиндром и жидкостью в момент начала импульсного движения цилиндра. Приведенные теоретические данные справедливы для любых чисел Прандтля и чисел Рейнольдса, больших 100. Полученные числа Нуссельта сравниваются с имеющимися численными и теоретическими результатами.